

Integrability of Vortex Equations on Riemann Surfaces

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Abstract

The Abelian Higgs model on a compact Riemann surface Σ of genus g is considered. We show that for $g > 1$ the Bogomolny equations for multi-vortices at critical coupling can be obtained as compatibility conditions of two linear equations (Lax pair) which are written down explicitly. These vortices correspond precisely to $SO(3)$ -symmetric Yang-Mills instantons on the (conformal) gravitational instanton $\Sigma \times S^2$ with a scalar-flat Kähler metric. Thus, the standard methods of constructing solutions and studying their properties by using Lax pairs (twistor approach, dressing method etc.) can be applied to the vortex equations on Σ . In the twistor description, solutions of the integrable vortex equations correspond to rank-2 holomorphic vector bundles over the complex 3-dimensional twistor space of $\Sigma \times S^2$. We show that in the general (nonintegrable) case there is a bijection between the moduli spaces of solutions to vortex equations on Σ and of pseudo-holomorphic bundles over the almost complex twistor space.

1 Introduction

The Abelian Yang-Mills-Higgs model on $\mathbb{R} \times \mathbb{R}^2$ at critical value of the coupling constant (the Bogomolny regime) admits static finite energy vortex solutions [1, 2] whose existence on \mathbb{R}^2 was proved by Taubes [3, 4]. They describe magnetic flux tubes (Abrikosov strings [1, 2]) penetrating a two-dimensional superconductor. Stability of vortices is ensured by topology [5]. The Taubes results for critically coupled vortices were generalized to compact Riemann surfaces Σ [6, 7]. In these investigations the main attention was devoted to study of Kähler geometry of the moduli space of vortices on Σ and to dynamics of vortices in Manton's adiabatic approximation (see e.g. [8]–[12]).

Recall that besides proving the existence theorem, Taubes has also shown that the standard vortex equations on \mathbb{R}^2 are equivalent to $\mathrm{SO}(3)$ -symmetric $\mathrm{SU}(2)$ self-dual Yang-Mills (SDYM) equations on $\mathbb{R}^2 \times S^2$ [4]. In other words, there is a one-to-one correspondence between $\mathrm{SO}(3)$ -equivariant¹ instantons on $\mathbb{R}^2 \times S^2$ for the gauge group $\mathrm{SU}(2)$ and vortices in the Abelian Higgs model² on \mathbb{R}^2 . This correspondence works also for vortices on any compact Riemann surface Σ of genus g : there is an equivalence between vortices (A, ϕ) on Σ and $\mathrm{SO}(3)$ -equivariant Yang-Mills instantons \mathcal{A} on $\Sigma \times S^2$, with the vortex number N (first Chern number of the $\mathrm{U}(1)$ connection A on the Hermitian line bundle E over Σ) equals the instanton number (the minus second Chern number of the symmetric connection \mathcal{A} on the rank-2 vector bundle \mathcal{E} over $\Sigma \times S^2$) [15]. We describe this correspondence in explicit form following [16]. In fact, we consider the pure $\mathrm{SU}(2)$ Yang-Mills action on $\Sigma \times S^2$ and show how it reduces to the action of the Abelian Higgs model on Σ . Then from the standard Bogomolny argument it follows that for a fixed vortex number $N \geq 0$ the minimum of the action functional is governed by solutions of the first-order vortex equations on Σ .

The above correspondence can be advanced further. Namely, recall that to each oriented Riemannian 4-dimensional manifold M one can associate a real 6-dimensional manifold \mathcal{Z} , the *twistor space* of M , which has a canonical almost complex structure \mathcal{J} [17]. For zero scalar curvature $R_M = R_\Sigma + R_{S^2}$ of $M = \Sigma \times S^2$, i.e. for³

$$R_\Sigma = -R_{S^2} = -\frac{2}{R^2}, \quad (1.1)$$

where R is the radius of S^2 , the almost complex structure \mathcal{J} is integrable and \mathcal{Z} becomes a complex 3-dimensional manifold [18]. Then we can pull back the instanton bundle \mathcal{E} over M to a holomorphic bundle $\hat{\mathcal{E}}$ over \mathcal{Z} [17, 21]. We show that in this (integrable) case the vortex equations on Σ can be obtained as the compatibility conditions of linear differential equations defining holomorphic sections of the bundle $\hat{\mathcal{E}} \rightarrow \mathcal{Z}$. Thus, we extend the correspondence between vortices on Σ and Yang-Mills instantons on $\Sigma \times S^2$ further and show that for $g > 1$ the vortex equations⁴ on Σ are equivalent to the equations defining an integrable holomorphic structure on the smooth complex vector bundle $\hat{\mathcal{E}}$ over the complex 3-dimensional twistor space \mathcal{Z} .

¹This means a generalized $\mathrm{SO}(3)$ -invariance, i.e. invariance up to gauge transformations [13]. Identifying $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ with $\mathbb{CP}^1 = \mathrm{SU}(2)/\mathrm{U}(1)$, we will also speak about $\mathrm{SU}(2)$ -equivariance.

²See also the previous work of Witten [14], who reduced the SDYM equations on \mathbb{R}^4 to vortex equations on the hyperbolic plane \mathbb{H}^2 .

³Such manifolds have self-dual Weyl tensor [18] and give particular examples of conformal gravitational instantons discussed e.g. in [17, 19, 20]. These instantons minimize the action quadratic in the Weyl tensor for the metric on M .

⁴The condition $g > 1$ is needed here since (1.1) can be satisfied only for Riemann surfaces of genus $g > 1$.

The picture is different for the cases $\Sigma = S^2$ ($g = 0$)⁵, $\Sigma = T^2$ ($g = 1$), Σ with $g > 1$ and $R_\Sigma \neq -R_{S^2}$ as well as for the noncompact case of \mathbb{R}^2 . In all these cases an almost complex structure on the twistor space \mathcal{Z} of $M = \Sigma \times S^2$ is *not integrable* as well as an almost complex structure on the vector bundle $\hat{\mathcal{E}} \rightarrow \mathcal{Z}$ which is the pull-back of an instanton bundle $\mathcal{E} \rightarrow M$. We argue that $\hat{\mathcal{E}}$ with the pulled-back connection $\hat{\mathcal{A}} = \pi^* \mathcal{A}$, for $\pi : \mathcal{Z} \rightarrow M$, has the structure of Bryant's *pseudo-holomorphic bundle* [24] with the curvature $\hat{\mathcal{F}}$ of type (1,1). Furthermore, we show that the vortex equations on Σ (including the case of \mathbb{R}^2) are equivalent to the SU(2)-equivariant *pseudo-Hermitian-Yang-Mills equations* on the almost complex twistor space \mathcal{Z} . However, the vortex equations in all these cases cannot be written as compatibility conditions of linear (Lax) equations. In the particular case of $g > 1$ and equality (1.1), all almost complex structures become integrable, Lax pair appears and the prefix “pseudo-” disappears.

2 Scalar-flat gravitational instantons and twistors

Here we consider Kähler metrics on a product $M = \Sigma_1 \times \Sigma_2$ of two Riemann surfaces of genera g_1 and g_2 and the twistor description of such four-dimensional Riemannian manifolds.

Riemann surfaces. Consider a compact Riemann surface Σ of genus $g \geq 0$ with the metric and the volume form given in local (conformal) coordinates w, \bar{w} by

$$ds^2 \equiv ds_\Sigma^2 = 2g_{w\bar{w}} dw d\bar{w} \quad (2.1)$$

and

$$\omega \equiv \omega_\Sigma = i g_{w\bar{w}} dw \wedge d\bar{w} . \quad (2.2)$$

Since Σ is a Kähler manifold, it follows that

$$\Gamma_{ww}^w = 2\partial_w \log \rho \quad \text{and} \quad \Gamma_{\bar{w}\bar{w}}^{\bar{w}} = 2\partial_{\bar{w}} \log \rho \quad \text{with} \quad \rho^2 := g_{w\bar{w}} , \quad (2.3)$$

$$R_{w\bar{w}} = -2\partial_w \partial_{\bar{w}} \log \rho = \varkappa g_{w\bar{w}} , \quad (2.4)$$

where Γ_{ww}^w and $\Gamma_{\bar{w}\bar{w}}^{\bar{w}}$ are nonvanishing components of the Christoffel symbols and $R_{w\bar{w}}$ is a component of the Ricci tensor.

For the scalar curvature of Σ we have

$$R_\Sigma = 2g^{w\bar{w}} R_{w\bar{w}} = 2\varkappa = \text{const.} \quad (2.5)$$

Note that for genus $g \neq 1$ the Riemann surface Σ has area equal to

$$\text{Vol}(\Sigma) = \int_\Sigma \omega_\Sigma = \frac{4\pi}{\varkappa} (1 - g) \quad (2.6)$$

which follows from (2.5) and the Gauss-Bonnet theorem.

Four-manifold M . Let us consider a smooth oriented real four-dimensional manifold M given by a product of two Riemann surfaces of genera g_1 and g_2 ,

$$M = \Sigma_1 \times \Sigma_2 , \quad (2.7)$$

⁵Note that the manifold $M = S^2 \times S^2$ with equal scalar curvatures of two-spheres is an Einstein manifold which is considered as a gravitational instanton in Euclidean quantum gravity, see e.g. [22, 23].

with the product metric

$$ds_M^2 = ds_1^2 + ds_2^2 = 2 g_{y_1 \bar{y}_1} dy_1 d\bar{y}_1 + 2 g_{y_2 \bar{y}_2} dy_2 d\bar{y}_2 \quad (2.8)$$

written in local complex coordinates on Σ_1 and Σ_2 , respectively.

We consider a principal bundle $P = P(M, \text{SO}(4))$ of orthonormal frames on a Riemannian 4-manifold M . For $M = \Sigma_1 \times \Sigma_2$ the holonomy group is reduced to $\text{U}(1) \times \text{U}(1) \subset \text{U}(2) \subset \text{SO}(4)$ and for the components $g^{y_1 \bar{y}_1} = 1/g_{y_1 \bar{y}_1}$ and $g^{y_2 \bar{y}_2} = 1/g_{y_2 \bar{y}_2}$ of inverse metric we have

$$g^{y_1 \bar{y}_1} = e_1^{y_1} e_1^{\bar{y}_1} \quad \text{and} \quad g^{y_2 \bar{y}_2} = e_2^{y_2} e_2^{\bar{y}_2} , \quad (2.9)$$

where $e_1^{y_1}, \dots, e_2^{\bar{y}_2}$ are unitary (local) frame fields. We also introduce a basis of type (1,0) and (0,1) vector fields

$$e_1 := e_1^{y_1} \partial_{y_1} , \quad e_2 := e_2^{y_2} \partial_{y_2} , \quad e_{\bar{1}} := e_1^{\bar{y}_1} \partial_{\bar{y}_1} \quad \text{and} \quad e_{\bar{2}} := e_2^{\bar{y}_2} \partial_{\bar{y}_2} , \quad (2.10)$$

which are sections of the complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ with TM associated to the principal bundle $P(M, \text{SO}(4))$. Dual basis of type (1,0) and (0,1) forms is

$$\beta^1 := e_1^1 dy_1 , \quad \beta^2 := e_2^2 dy_2 , \quad \beta^{\bar{1}} := e_{\bar{1}}^1 d\bar{y}_1 \quad \text{and} \quad \beta^{\bar{2}} := e_{\bar{2}}^2 d\bar{y}_2 , \quad (2.11)$$

where $e_{y_1}^1$ etc. are inverse to $e_1^{y_1}$ etc., i.e. $e_{y_1}^1 e_1^{y_1} = 1$ etc.

Introducing

$$\rho_1^2 := g_{y_1 \bar{y}_1}(y_1, \bar{y}_1) \quad \text{and} \quad \rho_2^2 := g_{y_2 \bar{y}_2}(y_2, \bar{y}_2) , \quad (2.12)$$

we obtain

$$\Gamma_{y_1 y_1}^{y_1} = 2 \partial_{y_1} \log \rho_1 , \quad \Gamma_{y_2 y_2}^{y_2} = 2 \partial_{y_2} \log \rho_2 , \quad \Gamma_{\bar{y}_1 \bar{y}_1}^{\bar{y}_1} = 2 \partial_{\bar{y}_1} \log \rho_1 , \quad \Gamma_{\bar{y}_2 \bar{y}_2}^{\bar{y}_2} = 2 \partial_{\bar{y}_2} \log \rho_2 , \quad (2.13)$$

$$R_{y_1 \bar{y}_1} = -\partial_{y_1} \partial_{\bar{y}_1} \log \rho_1^2 = \varkappa_1 g_{y_1 \bar{y}_1} \quad \text{and} \quad R_{y_2 \bar{y}_2} = -\partial_{y_2} \partial_{\bar{y}_2} \log \rho_2^2 = \varkappa_2 g_{y_2 \bar{y}_2} \quad (2.14)$$

with all other components vanishing. For the scalar curvature of M we have

$$R_M = 2g^{y_1 \bar{y}_1} R_{y_1 \bar{y}_1} + 2g^{y_2 \bar{y}_2} R_{y_2 \bar{y}_2} = 2(\varkappa_1 + \varkappa_2) . \quad (2.15)$$

Twistor space of M . Twistor space of an oriented four-dimensional manifold M can be defined as the associated bundle

$$\mathcal{Z} = P \times_{\text{SO}(4)} \mathbb{C}P^1 \quad (2.16)$$

with the canonical projection

$$\pi : \mathcal{Z} \rightarrow M . \quad (2.17)$$

Recall that P is the principal $\text{SO}(4)$ -bundle of oriented orthonormal frames. Fibres of the bundle (2.17) are two-spheres $S_x^2 \cong \mathbb{C}P_x^1 \cong \text{SO}(4)/\text{U}(2)$ parametrizing complex structures on tangent spaces $T_x M$ at $x \in M$.

Equivalent definition of the twistor bundle (2.17) can be obtained by considering the vector bundle $\Lambda^2 M$ of two-forms on M , associated to the principal bundle P . Using projectors on the subspaces of self-dual Λ_+^2 and anti-self-dual Λ_-^2 two-forms, one can split $\Lambda^2 M$ into the direct sum $\Lambda^2 M = \Lambda_+^2 M \oplus \Lambda_-^2 M$ of subbundles of self-dual and anti-self-dual two-forms on M . Then the twistor

space can be introduced as the unit sphere bundle $\mathcal{Z} = S_1(\Lambda_-^2 M)$ in the vector bundle $\Lambda_-^2 M$. That is why the Levi-Civita connection on P determines that on $\Lambda^2 M$ and induces a connection on \mathcal{Z} which is the anti-self-dual part $\Gamma_- = (\Gamma_-^i), i = 1, 2, 3$, of the Levi-Civita connection [17]. This connection generates the splitting of the tangent bundle $T\mathcal{Z}$ into the direct sum $T\mathcal{Z} = H \oplus V$, where $V = \text{Ker } \pi_*$ is the subbundle of vectors tangent to fibres $\mathbb{C}P^1$, and $H \cong T\mathcal{Z}/V$ consists of vectors horizontal with respect to the connection Γ_- on \mathcal{Z} . According to the canonical definition, the horizontal lift \tilde{X} of any vector field X on M is defined as

$$\tilde{X} := X + (X \lrcorner \Gamma_-^i) L_i, \quad (2.18)$$

where $X \lrcorner \Gamma_-^i$ denotes the interior product of a vector field and one-form, and L_i 's are vector fields on fibres $\mathbb{C}P^1 \hookrightarrow \mathcal{Z}$ which give a realization of the generators of the group $\text{SU}(2)$. By construction, \tilde{X} is a section of the bundle $H \rightarrow \mathcal{Z}$, X is a section of TM and $\pi_* \tilde{X} = X$.

We lift our frame vector fields (2.10) to \mathcal{Z} , switch to a complex basis by taking holomorphic parts of vector fields L_i and introduce type (0,1) vector fields on \mathcal{Z} as

$$V_1 := \tilde{e}_1 - \lambda \tilde{e}_2, \quad V_2 := \tilde{e}_2 + \lambda \tilde{e}_1 \quad \text{and} \quad V_3 := \partial_{\bar{\lambda}}, \quad (2.19)$$

where in local complex coordinates y_1, y_2, λ on \mathcal{Z} ($\lambda \in \mathbb{C}P^1 \hookrightarrow \mathcal{Z}$) we have

$$\tilde{e}_1 = \rho_1^{-1} \partial_{y_1} - \rho_1^{-1} (\partial_{y_1} \log \rho_1) \lambda \partial_{\lambda}, \quad \tilde{e}_2 = \rho_2^{-1} \partial_{y_2} - \rho_2^{-1} (\partial_{y_2} \log \rho_2) \lambda \partial_{\lambda}, \quad (2.20a)$$

$$\tilde{e}_{\bar{1}} = \rho_1^{-1} \partial_{\bar{y}_1} + \rho_1^{-1} (\partial_{\bar{y}_1} \log \rho_1) \lambda \partial_{\lambda} \quad \text{and} \quad \tilde{e}_{\bar{2}} = \rho_2^{-1} \partial_{\bar{y}_2} + \rho_2^{-1} (\partial_{\bar{y}_2} \log \rho_2) \lambda \partial_{\lambda}. \quad (2.20b)$$

Almost complex structure on \mathcal{Z} . The vector fields (2.19) define an *almost complex structure* \mathcal{J} on \mathcal{Z} such that

$$\mathcal{J}(V_{\bar{k}}) = -iV_k \quad (2.21)$$

for $k = 1, 2, 3$. In fact, the almost complex structure \mathcal{J} defined as above on the twistor space \mathcal{Z} of $M = \Sigma_1 \times \Sigma_2$ is canonical and does not depend on the choice of local coordinates [17]. For commutators of type (0,1) vector fields (2.19) we have

$$[V_1, V_2] = \lambda \rho_1^{-2} (\partial_{y_1} \rho_1) V_1 + \lambda \rho_2^{-2} (\partial_{y_2} \rho_2) V_2 + 2\lambda^2 (\varkappa_1 + \varkappa_2) V_3 \quad \text{and} \quad [V_1, V_3] = 0 = [V_2, V_3], \quad (2.22)$$

where $V_3 = \partial_{\lambda}$ is the (1,0) vector field.

Recall that for integrability of an almost complex structure \mathcal{J} on \mathcal{Z} it is necessary and sufficient that the commutator of any two vector fields of type (0,1) w.r.t. \mathcal{J} is of type (0,1). For our case we see from (2.22) that \mathcal{J} is integrable - and \mathcal{Z} is a complex manifold - if and only if

$$\varkappa_1 = -\varkappa_2, \quad (2.23)$$

i.e when the scalar curvature (2.15) of the Kähler manifold $\Sigma_1 \times \Sigma_2$ vanishes. Besides the case $g_1 = 1 = g_2$ (tori) this can be satisfied when one takes $g := g_1 \geq 2$ and $g_2 = 0$ (two-sphere).

Gravitational instantons. Recall that the Weyl tensor for the manifold $\Sigma_1 \times \Sigma_2$ with equal and opposite scalar curvatures of Σ_1 and Σ_2 is self-dual (or anti-self-dual for the inverse orientation) [18, 25]. Such manifolds $\Sigma_1 \times \Sigma_2$ with $\varkappa_1 = -\varkappa_2$ are considered as gravitational instantons in conformal gravity [17, 20]. However, the case $\varkappa_1 = \varkappa_2$ is also of interest since such 4-manifolds $\Sigma_1 \times \Sigma_2$ are smooth Einstein spaces which were also considered as gravitational instantons in Euclidean

quantum gravity (see e.g. [22, 23]). Note that the twistor space \mathcal{Z} of such manifolds is an almost complex manifold with the nonintegrable almost complex structure \mathcal{J} defined by formulae (2.19)-(2.22). In what follows we consider the case with

$$\kappa_1 = \kappa \in \mathbb{R} \quad \text{and} \quad \kappa_2 = \frac{1}{R^2} > 0 \quad (2.24)$$

including $\kappa = 0$ ($T^2 \times S^2$) and the special cases $\kappa_1 = -\kappa_2$ ($\Sigma \times S^2$) and $\kappa_1 = \kappa_2$ ($S^2 \times S^2$) corresponding to both types of (conformally self-dual and non-self-dual) gravitational instantons.

3 Vortices on Σ as Yang-Mills instantons on $\Sigma \times \mathbb{C}P^1$

Riemann sphere. Let $\Sigma_2 = \mathbb{C}P^1 \cong S^2$ be the standard two-sphere of constant radius R . In local coordinates on $\mathbb{C}P^1$ the metric reads

$$ds_2^2 = 2 g_{y\bar{y}} dy d\bar{y} = \frac{4R^4}{(R^2 + y\bar{y})^2} dy d\bar{y} = R^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (3.1)$$

for

$$y := y_2 = R \tan\left(\frac{\theta}{2}\right) \exp(-i\varphi), \quad \bar{y} := \bar{y}_2 = R \tan\left(\frac{\theta}{2}\right) \exp(i\varphi), \quad 0 \leq \theta < \pi, \quad 0 \leq \varphi \leq 2\pi \quad (3.2)$$

Note that (3.2) corresponds to the choice of orientation on S^2 inverse to the canonical one, i.e. $y_2 = x^3 - ix^4$ and $\bar{y}_2 = x^3 + ix^4$ for real local coordinates x^3, x^4 on S^2 . That is why for the volume form we have

$$\omega_2 = R^2 \sin^2 \theta d\theta \wedge d\varphi = -\frac{2i R^4}{(R^2 + y\bar{y})^2} dy \wedge d\bar{y} = -i g_{y\bar{y}} dy \wedge d\bar{y}, \quad (3.3)$$

i.e. ω_2 has the inverse sign in comparison with (2.2).

For the two-sphere

$$R_{y\bar{y}} = -\partial_y \partial_{\bar{y}} \log \rho_2^2 = \frac{1}{R^2} \rho_2^2 \quad \Rightarrow \quad \kappa_2 = \frac{1}{R^2} > 0, \quad (3.4)$$

and we have

$$\tilde{e}_2 = \frac{(R^2 + y\bar{y})}{\sqrt{2} R^2} \partial_y + \frac{\bar{y}}{\sqrt{2} R^2} \lambda \partial_\lambda \quad \text{and} \quad \tilde{e}_{\bar{2}} = \frac{(R^2 + y\bar{y})}{\sqrt{2} R^2} \partial_{\bar{y}} - \frac{y}{\sqrt{2} R^2} \lambda \partial_\lambda. \quad (3.5)$$

For the area of S^2 we have

$$\text{Vol}(S^2) = \int_{S^2} \omega_2 = 4\pi R^2, \quad (3.6)$$

which agrees with the general formula (2.6).

Monopole bundles. Consider the Hermitian line bundle $\mathcal{L} \rightarrow \mathbb{C}P^1$ (one-monopole bundle) with a unique $\text{SU}(2)$ -equivariant unitary connection

$$a = -\frac{i}{2} (1 - \cos \theta) d\varphi = \frac{1}{2(R^2 + y\bar{y})} (\bar{y} dy - y d\bar{y}). \quad (3.7)$$

Then on the bundle $\mathcal{L}^n := (\mathcal{L})^{\otimes n}$ with $n \in \mathbb{Z}$ we have⁶

$$a^{(n)} = n a \quad \text{and} \quad f^{(n)} := da^{(n)} = -\frac{nR^2}{(R^2 + y\bar{y})^2} dy \wedge d\bar{y} = \frac{n}{2iR^2} \omega_2. \quad (3.8)$$

The topological charge of this gauge field configuration is given by the first Chern number (equivalently the degree) of the complex line bundle \mathcal{L} ,

$$c_1(\mathcal{L}^n) = \frac{i}{2\pi} \int_{\mathbb{CP}^1} f^{(n)} = \frac{n}{4\pi R^2} \int_{\mathbb{CP}^1} \omega_2 = n \quad (3.9)$$

This configuration describes $|n|$ Dirac monopoles ($n > 0$) or antimonopoles ($n < 0$) sitting on the top of each other.

SU(2)-equivariant gauge potential. Let $\mathcal{E} \rightarrow M$ be a rank-2 SU(2)-equivariant complex vector bundle over $M = \Sigma \times \mathbb{CP}^1$ with the group SU(2) acting trivially on Σ and in the standard way by SU(2)-isometry on $\mathbb{CP}^1 = \text{SU}(2)/\text{U}(1)$. Let \mathcal{A} be an $su(2)$ -valued equivariant connection on \mathcal{E} . The explicit form of such a connection is known (see e.g. [13, 15, 16]); it has the form

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2}A \otimes 1 + 1 \otimes a & \frac{1}{\sqrt{2}}\phi \otimes \bar{\beta} \\ -\frac{1}{\sqrt{2}}\bar{\phi} \otimes \beta & -\frac{1}{2}A \otimes 1 - 1 \otimes a \end{pmatrix} \quad (3.10)$$

where A is an Abelian connection on the (Hermitian) complex line bundle E over the genus g compact Riemann surface Σ , a is the monopole connection (3.7) on the line bundle $\mathcal{L} \rightarrow \mathbb{CP}^1$, ϕ is a section of the bundle E , $\bar{\phi}$ is its complex conjugate and

$$\beta := \beta^2 \frac{\sqrt{2} R^2 dy}{R^2 + y\bar{y}} \quad \text{and} \quad \bar{\beta} := \beta^{\bar{2}} = \frac{\sqrt{2} R^2 d\bar{y}}{R^2 + y\bar{y}} \quad (3.11)$$

are forms on \mathbb{CP}^1 of type (1,0) and (0,1) given by (2.11). In local complex coordinates $z := y_1 = x^1 + ix^2$ and $\bar{z} := \bar{y}_1 = x^1 - ix^2$, we have $A = A(z, \bar{z})$ and $\phi = \phi(z, \bar{z})$.

Note that for any $\text{rank}(\mathcal{E}) \geq 2$ the \mathbb{CP}^1 -dependence is uniquely determined by the SU(2)-equivariance, by rank of \mathcal{E} and the monopole configuration on \mathbb{CP}^1 [15, 16]. In particular, in the above case of $\text{rank}(\mathcal{E}) = 2$ the forms (3.11) are the unique SU(2)-invariant (1,0) and (0,1) forms such that

$$d\bar{\beta} + 2a \wedge \bar{\beta} = 0 = d\beta - 2a \wedge \beta \quad (3.12)$$

and the Kähler (1,1)-form ω_2 on \mathbb{CP}^1 is $-i\beta \wedge \bar{\beta}$. The forms β and $\bar{\beta}$ take values in the bundles $K = \mathcal{L}^2$ and $K^{-1} = \mathcal{L}^{-2}$, respectively.

Field strength tensor. In local complex coordinates on $\Sigma \times \mathbb{CP}^1$ the calculation of the curvature \mathcal{F} for \mathcal{A} of the form (3.10) yields

$$\begin{aligned} \mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} &= \begin{pmatrix} \frac{1}{2}F - \frac{1}{2}(\frac{1}{R^2} - \phi\bar{\phi})\beta \wedge \bar{\beta} & \frac{1}{\sqrt{2}}(d\phi + A\phi) \wedge \bar{\beta} \\ -\frac{1}{\sqrt{2}}(d\bar{\phi} - A\bar{\phi}) \wedge \beta & -\frac{1}{2}F + \frac{1}{2}(\frac{1}{R^2} - \phi\bar{\phi})\beta \wedge \bar{\beta} \end{pmatrix} = \\ &= \mathcal{F}_{z\bar{z}} dz \wedge d\bar{z} + \mathcal{F}_{zy} dz \wedge dy + \mathcal{F}_{z\bar{y}} dz \wedge d\bar{y} + \mathcal{F}_{\bar{z}y} d\bar{z} \wedge dy + \mathcal{F}_{\bar{z}\bar{y}} d\bar{z} \wedge d\bar{y} + \mathcal{F}_{y\bar{y}} dy \wedge d\bar{y} \end{aligned} \quad (3.13)$$

⁶For more detailed description with transition functions etc. see e.g. [26].

with the non-vanishing field strength components

$$\mathcal{F}_{z\bar{z}} = \frac{1}{2} F_{z\bar{z}} \sigma_3, \quad \mathcal{F}_{y\bar{y}} = -\frac{\rho_2^2}{2} \left(\frac{1}{R^2} - \phi\bar{\phi} \right) \sigma_3, \quad (3.14)$$

$$\mathcal{F}_{\bar{z}y} = \frac{\rho_2}{\sqrt{2}} (\partial_{\bar{z}}\phi + A_{\bar{z}}\phi) \sigma_+, \quad \mathcal{F}_{zy} = \frac{\rho_2}{\sqrt{2}} (\partial_z\phi + A_z\phi) \sigma_+, \quad (3.15)$$

$$\mathcal{F}_{zy} = -\frac{\rho_2}{\sqrt{2}} (\partial_z\bar{\phi} - A_z\bar{\phi}) \sigma_-, \quad \mathcal{F}_{\bar{z}y} = -\frac{\rho_2}{\sqrt{2}} (\partial_{\bar{z}}\bar{\phi} - A_{\bar{z}}\bar{\phi}) \sigma_-, \quad (3.16)$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.17)$$

In (3.14) we have defined $F = dA = F_{z\bar{z}} dz \wedge d\bar{z} = (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) dz \wedge d\bar{z}$ for $A = A_z dz + A_{\bar{z}} d\bar{z}$.

Reduction of the Yang-Mills functional. The dimensional reduction of the Euclidean Yang-Mills equations from $\Sigma \times \mathbb{C}P^1$ to Σ can be seen at the level of the Yang-Mills Lagrangian. Substituting (3.14)-(3.16) into the standard Yang-Mills functional and performing the integral over $\mathbb{C}P^1$, we arrive at the action

$$\begin{aligned} S &= -\frac{1}{8\pi^2} \int_M \text{tr}(\mathcal{F} \wedge * \mathcal{F}) = -\frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{C}P^1} d^4x \sqrt{\det(g_{\rho\sigma})} \text{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) = \\ &= \frac{R^2}{2\pi} \int_{\Sigma} \omega_{\Sigma} \left\{ (g^{z\bar{z}})^2 (F_{z\bar{z}})^2 + 2g^{z\bar{z}} (D_z\phi D_{\bar{z}}\bar{\phi} + D_{\bar{z}}\phi D_z\bar{\phi}) + \left(\frac{1}{R^2} - \phi\bar{\phi} \right)^2 \right\}, \end{aligned} \quad (3.18)$$

where

$$\omega_{\Sigma} \equiv \omega_1 = i g_{z\bar{z}} dz \wedge d\bar{z} \quad (3.19)$$

and x^{μ} are real local coordinates⁷ on $M = \Sigma \times \mathbb{C}P^1$, $\mu, \nu, \dots = 1, \dots, 4$.

For the functional (3.18), using the standard Bogomolny arguments [27], one can obtain the inequality

$$S = \frac{R^2}{2\pi} \int_{\Sigma} i dz \wedge d\bar{z} \left\{ g^{z\bar{z}} \left(F_{z\bar{z}} + g_{z\bar{z}} \left(\phi\bar{\phi} - \frac{1}{R^2} \right) \right)^2 + 2D_{\bar{z}}\phi \overline{(D_z\phi)} \right\} + \frac{i}{2\pi} \int_{\Sigma} F \geq N, \quad (3.20)$$

where

$$N = -c_2(\mathcal{E}) = -\frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{C}P^1} \text{tr}(\mathcal{F} \wedge \mathcal{F}) = \frac{i}{2\pi} \int_{\Sigma} F = c_1(E) \quad (3.21)$$

is the vortex number. In the derivation of (3.20) it is assumed that $N \geq 0$ and similar inequality can be obtained for $N \leq 0$. Thus, the (minus) second Chern number of the $SU(2)$ -equivariant connection \mathcal{A} on the rank-2 bundle \mathcal{E} over $\Sigma \times \mathbb{C}P^1$ equals to the first Chern number of the connection A on the line bundle E over Σ .

Vortex equations on Σ . If $N \geq 0$, then (3.20) is an equality if and only if the Yang-Mills field \mathcal{F} on $\Sigma \times \mathbb{C}P^1$ is self-dual,

$$*\mathcal{F} = \mathcal{F}, \quad (3.22)$$

⁷One can take e.g. $y_1 = z = x^1 + ix^2$ and $y_2 = y = x^3 - ix^4$

and substitution of (3.14)-(3.16) shows that Eqs. (3.22) are equivalent to the vortex equations on Σ ,

$$F_{z\bar{z}} = g_{z\bar{z}} \left(\frac{1}{R^2} - \phi\bar{\phi} \right) \quad \Leftrightarrow \quad iF = \left(\frac{1}{R^2} - \phi\bar{\phi} \right) \omega_\Sigma, \quad (3.23a)$$

$$\partial_{\bar{z}}\phi + A_{\bar{z}}\phi = 0 \quad \Leftrightarrow \quad \bar{\partial}_A\phi = 0. \quad (3.23b)$$

Note that the case $N \leq 0$ corresponds to the anti-self-dual Yang-Mills equations $*\mathcal{F} = -\mathcal{F}$ which reduce to the anti-vortex equations

$$F_{z\bar{z}} = -g_{z\bar{z}} \left(\frac{1}{R^2} - \phi\bar{\phi} \right) \quad \text{and} \quad \partial_z\phi + A_z\phi = 0. \quad (3.24)$$

For this reduction one should simply change $y \rightarrow \bar{y}$ in (3.10) and (3.11) which is equivalent to the change of the orientation $x^4 \rightarrow -x^4$ of M . It is well known that self-duality is transformed into anti-self-duality under the change of orientation on M .

From (3.21) we see that the instanton number of the $SU(2)$ -equivariant Yang-Mills field on $\Sigma \times \mathbb{C}P^1$ is equal to the number $N \geq 0$ of vortices on the Riemann surface Σ . Furthermore, for $g \neq 1$ from (3.23) it follows that

$$\frac{i}{2\pi} \int_\Sigma F + \int_\Sigma \phi\bar{\phi}\omega_\Sigma = \frac{1}{2\pi R^2} \int_\Sigma \omega_\Sigma = \frac{2}{\varkappa R^2} (1 - g) \quad (3.25)$$

and we obtain (cf. [6]) the inequality

$$N \leq \frac{2}{\varkappa R^2} (1 - g). \quad (3.26)$$

For $g = 1$ we have $N \leq \text{Vol}(T^2)/2\pi R^2$. Recall that in our derivation of the vortex equations (3.23) the parameter $\varkappa = \frac{1}{2}R_\Sigma$ is not fixed. In particular, one can take $g = 0$ and $g = 1$ and obtains vortices on the sphere S^2 or torus T^2 .

4 Twistor description of vortex equations

In Section 3 we discussed the relation between vortices on Riemann surfaces Σ and Yang-Mills instantons on the manifolds $\Sigma \times \mathbb{C}P^1$. Here, we push this correspondence further and show that to any solution (A, ϕ) of vortex equations on Σ there corresponds a connection $\hat{\mathcal{A}}$ defining a pseudo-holomorphic structure on a rank-2 complex vector bundle $\hat{\mathcal{E}}$ over an almost complex twistor space \mathcal{Z} for $\Sigma \times \mathbb{C}P^1$. We also discuss the integrability of this pseudo-holomorphic structure and the corresponding vortex equations.

Pull-back to the twistor space. Consider an $SU(2)$ -equivariant rank-2 complex vector bundle $\mathcal{E} \rightarrow \Sigma \times \mathbb{C}P^1$ with a connection \mathcal{A} described in Section 3. Using the projection (2.17), we pull \mathcal{E} back to a bundle $\hat{\mathcal{E}} := \pi^*\mathcal{E}$ over \mathcal{Z} :

$$\begin{array}{ccc} \hat{\mathcal{E}} & \longrightarrow & \mathcal{Z} \\ & \downarrow \pi & \\ \mathcal{E} & \longrightarrow & M \end{array} \quad (4.1)$$

In accordance with the definition of the pull-back, the connection $\hat{\mathcal{A}} := \pi^* \mathcal{A}$ on $\hat{\mathcal{E}}$ is flat along the fibres $\mathbb{C}P_x^1$ of the bundle $\pi : \mathcal{Z} \rightarrow M$ and we can set the components $\hat{\mathcal{A}}_\lambda$ and $\hat{\mathcal{A}}_{\bar{\lambda}}$ of the restriction of $\hat{\mathcal{A}}$ to $\mathbb{C}P_x^1 \hookrightarrow \mathcal{Z}$ equal to zero. Thus, restrictions of smooth vector bundle $\hat{\mathcal{E}}$ to fibres $\mathbb{C}P_x^1$ of the projection π are naturally holomorphic and holomorphically trivial for each $x \in M = \Sigma \times \mathbb{C}P^1$.

Note that a lift of the generators of the group $SU(2)$ acting on $\mathbb{C}P^1 = SU(2)/U(1)$ to \mathcal{Z} can be defined analogously to the flat Euclidean case [28]. Namely, we lift the $SU(2)$ -generators to \mathcal{Z} so that an almost complex structure on \mathcal{Z} is invariant under the action of the group $SU(2)$ generated by these lifted vector fields. Then one can impose $SU(2)$ -equivariance conditions on the vector bundle $\hat{\mathcal{E}} \rightarrow \mathcal{Z}$ and connection $\hat{\mathcal{A}}$ on $\hat{\mathcal{E}}$.

Connection along $T^{0,1}\mathcal{Z}$. Recall that the twistor space \mathcal{Z} of M has a canonical almost complex structure \mathcal{J} [17]. In the considered case it is defined by the type (0,1) vector fields (2.19). Hence, we can introduce a (0,1) part $\hat{\nabla}^{0,1}$ of the covariant derivative $\hat{\nabla}$ on $\hat{\mathcal{E}}$ by formulae

$$\hat{\nabla}_{V_{\bar{1}}} \equiv V_{\bar{1}} + \hat{\mathcal{A}}_{V_{\bar{1}}} := \tilde{e}_{\bar{1}} - \lambda \tilde{e}_2 + \mathcal{A}_{\bar{1}} - \lambda \mathcal{A}_2, \quad (4.2a)$$

$$\hat{\nabla}_{V_{\bar{2}}} \equiv V_{\bar{2}} + \hat{\mathcal{A}}_{V_{\bar{2}}} := \tilde{e}_{\bar{2}} + \lambda \tilde{e}_1 + \mathcal{A}_{\bar{2}} + \lambda \mathcal{A}_1, \quad (4.2b)$$

$$\hat{\nabla}_{V_{\bar{3}}} \equiv V_{\bar{3}} + \hat{\mathcal{A}}_{V_{\bar{3}}} := \partial_{\bar{\lambda}}, \quad (4.3)$$

where $\hat{\nabla}_X$ denotes the covariant derivative along the vector field X . The components \mathcal{A}_1 etc. can be extracted from (3.10),

$$\mathcal{A}_1 = \frac{1}{2} A_1 \sigma_3, \quad \mathcal{A}_{\bar{1}} = \frac{1}{2} A_{\bar{1}} \sigma_3, \quad \mathcal{A}_2 = a_2 \sigma_3 - \frac{\bar{\phi}}{\sqrt{2}} \sigma_-, \quad \text{and} \quad \mathcal{A}_{\bar{2}} = a_{\bar{2}} \sigma_3 + \frac{\phi}{\sqrt{2}} \sigma_+, \quad (4.4)$$

where $\mathcal{A}_1 = e_1^z \mathcal{A}_z$, $\mathcal{A}_{\bar{1}} = e_{\bar{1}}^{\bar{z}} \mathcal{A}_{\bar{z}}$, $\mathcal{A}_2 = e_2^y \mathcal{A}_y$, $\mathcal{A}_{\bar{2}} = e_{\bar{2}}^{\bar{y}} \mathcal{A}_{\bar{y}}$ and similarly for A_1 , $A_{\bar{1}}$ and a_2 , $a_{\bar{2}}$.

Pseudo-holomorphic bundles. Let us consider an almost complex manifold (Q, \mathcal{J}) and a complex vector bundle \mathcal{V} over Q endowed with a connection $\hat{\mathcal{A}}$. According to Bryant [24], a connection $\hat{\mathcal{A}}$ on \mathcal{V} is said to define a pseudo-holomorphic structure on \mathcal{V} if it has curvature $\hat{\mathcal{F}}$ of type (1,1). In principle, one can also endow \mathcal{V} with a Hermitian metric and choose $\hat{\mathcal{A}}$ to be compatible with the Hermitian structure on \mathcal{V} [24]. If, in addition, η is an almost Hermitian metric on (Q, \mathcal{J}) and

$$\text{tr}_\eta(\hat{\mathcal{F}}) = i\gamma \text{Id}_{\hat{\mathcal{E}}} \quad \text{with} \quad \gamma \in \mathbb{R}, \quad (4.5)$$

the connection $\hat{\mathcal{A}}$ is said to be pseudo-Hermitian-Yang-Mills. Here, we will consider the equations⁸ (4.5) with $\gamma = 0$.

Consider the $SU(2)$ -equivariant complex vector bundle $\hat{\mathcal{E}} = \pi^* \mathcal{E}$ from (4.1) with a connection $\hat{\mathcal{A}} = \pi^* \mathcal{A}$. In [17], it was shown that pulling-back a real structure $\tau : \mathcal{Z} \rightarrow \hat{\mathcal{E}}$, one can endow $\hat{\mathcal{E}}$ with a Hermitian structure. Then pseudo-holomorphicity of the bundle $\hat{\mathcal{E}}$ is equivalent to the equations

$$\hat{\mathcal{F}}^{0,2} = 0 \quad \Leftrightarrow \quad \hat{\mathcal{F}}(V_i, V_j) = [\hat{\nabla}_{V_i}, \hat{\nabla}_{V_j}] - \hat{\nabla}_{[V_i, V_j]} = 0, \quad (4.6)$$

where $\hat{\nabla}_{V_i}$ for $i = 1, 2, 3$ are given in (4.2) and (4.3).

⁸Equations (4.5) together with $\hat{\mathcal{F}}^{0,2} = 0$ can also be called pseudo-Donaldson-Uhlenbeck-Yau equations (cf. [29]) with the prefix “pseudo-”, since Q is not a Kähler and not even a complex manifold.

Pseudo-holomorphicity and instantons. After simple calculations we see that the only nontrivial components of the tensor $\hat{\mathcal{F}}^{0,2}$ reads

$$\hat{\mathcal{F}}(V_1, V_2) = \mathcal{F}_{\bar{1}\bar{2}} - \lambda(\mathcal{F}_{1\bar{1}} + \mathcal{F}_{2\bar{2}}) + \lambda^2 \mathcal{F}_{12} , \quad (4.7)$$

where

$$\mathcal{F}_{\bar{1}\bar{2}} = e_{\bar{1}}\mathcal{A}_{\bar{2}} - e_{\bar{2}}\mathcal{A}_{\bar{1}} + [\mathcal{A}_{\bar{1}}, \mathcal{A}_{\bar{2}}] = e_{\bar{1}}^{\bar{z}}e_{\bar{2}}^{\bar{y}}\mathcal{F}_{\bar{z}\bar{y}} , \quad (4.8a)$$

$$\mathcal{F}_{12} = -(\mathcal{F}_{\bar{1}\bar{2}})^\dagger = e_1\mathcal{A}_2 - e_2\mathcal{A}_1 + [\mathcal{A}_1, \mathcal{A}_2] = e_1^ze_2^y\mathcal{F}_{zy} , \quad (4.8b)$$

$$\mathcal{F}_{1\bar{1}} = e_1\mathcal{A}_{\bar{1}} - e_{\bar{1}}\mathcal{A}_1 + [\mathcal{A}_1, \mathcal{A}_{\bar{1}}] - \rho_1^{-1}(e_{\bar{1}}\rho_1)\mathcal{A}_1 + \rho_1^{-1}(e_1\rho_1)\mathcal{A}_{\bar{1}} = g^{z\bar{z}}\mathcal{F}_{z\bar{z}} , \quad (4.8c)$$

$$\mathcal{F}_{2\bar{2}} = e_2\mathcal{A}_{\bar{2}} - e_{\bar{2}}\mathcal{A}_2 + [\mathcal{A}_2, \mathcal{A}_{\bar{2}}] - \rho_2^{-1}(e_{\bar{2}}\rho_2)\mathcal{A}_2 + \rho_2^{-1}(e_2\rho_2)\mathcal{A}_{\bar{2}} = g^{y\bar{y}}\mathcal{F}_{y\bar{y}} . \quad (4.8d)$$

Thus, we see that equations (4.6) on \mathcal{Z} , defining pseudo-holomorphic structure on $\hat{\mathcal{E}}$, reduce to the equation

$$\hat{\mathcal{F}}(V_1, V_2) = \mathcal{F}_{\bar{1}\bar{2}} - \lambda(\mathcal{F}_{1\bar{1}} + \mathcal{F}_{2\bar{2}}) + \lambda^2 \mathcal{F}_{12} = 0 , \quad (4.9)$$

which is equivalent to the SDYM equations on $\Sigma \times \mathbb{CP}^1$,

$$\mathcal{F}_{\bar{z}\bar{y}} = 0 , \quad g^{z\bar{z}}\mathcal{F}_{z\bar{z}} + g^{y\bar{y}}\mathcal{F}_{y\bar{y}} = 0 \quad \text{and} \quad \mathcal{F}_{zy} = 0 . \quad (4.10)$$

Conversely, every pseudo-holomorphic $\text{SU}(2)$ -equivariant vector bundle $\hat{\mathcal{E}}$ over \mathcal{Z} such that it is holomorphically trivial on each $\mathbb{CP}_x^1 \hookrightarrow \mathcal{Z}$, $x \in \Sigma \times \mathbb{CP}^1$, is the pull-back to \mathcal{Z} of an $\text{SU}(2)$ -equivariant bundle \mathcal{E} with a self-dual connection on $\Sigma \times \mathbb{CP}^1$.⁹ Recall that equations (4.10) are equivalent in turn to vortex equations on Σ as discussed in Section 3.

Pseudo-Hermitian-Yang-Mills equations on \mathcal{Z} . One can check by direct calculation that the connection $\hat{\mathcal{A}} = \pi^*\mathcal{A}$ satisfies the equations

$$\hat{\mathcal{F}}(V_1, V_1) + \hat{\mathcal{F}}(V_2, V_2) + \hat{\mathcal{F}}(V_3, V_3) = 0 , \quad (4.11)$$

which also reduce to the SDYM equations (4.10). Together, (4.9) and (4.11) constitute pseudo-Hermitian-Yang-Mills equations on \mathcal{Z} . However, in our concrete case of twistor space \mathcal{Z} equation (4.11) does not impose additional restrictions on $\hat{\mathcal{A}}$ in comparison with (4.9). This is in conformity with [24].

Summarizing our above discussion, we have established the diagram

$$\begin{array}{ccc} & \text{pseudo-Hermitian-Yang-Mills} \\ & \text{equations on the twistor} \\ & \text{space } \mathcal{Z} \text{ of } \Sigma \times \mathbb{CP}^1 \\ \swarrow \nearrow & & \nwarrow \searrow \\ \text{vortex equations on } \Sigma & \longleftrightarrow & \text{Yang-Mills instanton} \\ & & \text{equations on } \Sigma \times \mathbb{CP}^1 \end{array} \quad (4.12)$$

describing equivalent theories defined on different spaces. Furthermore, there are bijections between the moduli spaces of solutions to all three types of equations mentioned in (4.12).

⁹This can be straightforwardly generalized to a correspondence between Hermitian vector bundles \mathcal{E} with self-dual connections on an arbitrary oriented 4-manifold M and pseudo-holomorphic bundles $\hat{\mathcal{E}}$ over an almost complex twistor space \mathcal{Z} of M . This generalization of the Theorem 5.2 in [17] will be considered elsewhere.

Integrable case. In the general case of the twistor space \mathcal{Z} of $M = \Sigma \times \mathbb{CP}^1$, the subbundle $T^{0,1}\mathcal{Z}$ of $T^{\mathbb{C}}\mathcal{Z}$ is not integrable since the distribution of vector fields of type (0,1) is not closed under the Lie bracket. However, for Σ with genus $g \geq 2$, one can always rescale the metric on Σ to fulfill the condition

$$\varkappa = -\frac{1}{R^2} \quad \Rightarrow \quad R_M = R_\Sigma + R_{S^2} = 0 . \quad (4.13)$$

In this case the almost complex structure \mathcal{J} on the twistor space \mathcal{Z} of M becomes integrable - distribution of (0,1) vector fields (2.19) is closed under the Lie bracket - and the pseudo-holomorphic bundle $\hat{\mathcal{E}} \rightarrow \mathcal{Z}$ becomes holomorphic. In other words, the almost complex structure on $\hat{\mathcal{E}}$ defined by (0,1)-type connection (4.2) and (4.3) becomes integrable and one can introduce holomorphic sections of $\hat{\mathcal{E}}$.

Linear system. Recall that a (local) section χ of the complex vector bundle $\hat{\mathcal{E}}$ is said to be holomorphic if

$$\hat{\nabla}^{0,1}\chi = 0 \quad \Leftrightarrow \quad \hat{\nabla}_{V_1}\chi = \hat{\nabla}_{V_2}\chi = \hat{\nabla}_{V_3}\chi = 0 . \quad (4.14)$$

Accordingly, the bundle $\hat{\mathcal{E}} \rightarrow \mathcal{Z}$ is said to be holomorphic if equations (4.14) are compatible, i.e. the (0,2) components of the curvature $\hat{\mathcal{F}}$ of the bundle $\hat{\mathcal{E}}$ vanish. This condition yields equations (4.6) (and (4.10)) but now with vector fields V_i which are closed under the Lie bracket.

Let us now introduce a 2×2 matrix ψ of fundamental solutions of eqs. (4.14), i.e. such that columns of ψ form smooth frame fields for $\hat{\mathcal{E}}$. Then we obtain two linear equations (Lax pair)

$$\hat{\nabla}_{V_1}\psi(x, \lambda) \equiv (V_1 + \hat{\mathcal{A}}_{V_1})\psi = 0 , \quad (4.15a)$$

$$\hat{\nabla}_{V_2}\psi(x, \lambda) \equiv (V_2 + \hat{\mathcal{A}}_{V_2})\psi = 0 , \quad (4.15b)$$

since the third linear equation $\hat{\nabla}_{V_3}\psi = 0$ is trivially solved for ψ which does not depend on $\bar{\lambda}$. Here $\lambda \in U_+ = \mathbb{CP}^1 \setminus \{\infty\} \subset \mathbb{CP}^1$.

Explicit Lax pair. In our case, the explicit form of the Lax pair (4.15) reads

$$\left[\tilde{e}_1 + \frac{1}{2} A_1 \sigma_3 - \lambda (\tilde{e}_2 + a_2 \sigma_3 - \frac{\bar{\phi}}{\sqrt{2}} \sigma_-) \right] \psi = 0 , \quad (4.16a)$$

$$\left[\tilde{e}_1 + \frac{1}{2} A_1 \sigma_3 + \frac{1}{\lambda} (\tilde{e}_2 + a_2 \sigma_3 + \frac{\phi}{\sqrt{2}} \sigma_+) \right] \psi = 0 , \quad (4.16b)$$

where $\tilde{e}_1, \tilde{e}_2, \tilde{e}_1$ and \tilde{e}_2 are written down in (2.20) and (3.5), and a_2, a_2 are given in (3.7). By direct calculation, one can see that the compatibility conditions of the linear equations (4.16),

$$[\hat{\nabla}_{V_1}, \hat{\nabla}_{V_2}]\psi = 0 , \quad (4.17)$$

are equivalent to the vortex equations (3.23) on Σ . Note that for $\text{span}\{V_i\}$ closed under the Lie bracket, (4.17) is equivalent to (4.6) but this is not true for the nonintegrable distribution $T^{0,1}\mathcal{Z}$.

Riemann-Hilbert problems. Let us consider an open subset \mathcal{U} of the manifold $\Sigma \times S^2$, restrict $\mathcal{Z}|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{CP}^1$ and consider a two-set open covering $\{U_+, U_-\}$ of the fibre \mathbb{CP}^1 in $\pi : \mathcal{U} \times \mathbb{CP}^1 \rightarrow \mathcal{U}$, $U_+ = \mathbb{CP}^1 \setminus \{\infty\}$ and $U_- = \mathbb{CP}^1 \setminus \{0\}$. Then the restriction of $\hat{\mathcal{E}}$ to $\mathcal{Z}|_{\mathcal{U}}$ will be described by a transition 2×2 matrix f_{+-} on $\mathcal{U} \times U_+ \cap U_-$, holomorphic in λ , whose restriction to $U_+ \cap U_- \hookrightarrow \mathcal{Z}|_{\mathcal{U}}$ is splitted,

$$f_{+-} = \psi_+^{-1}(x, \lambda) \psi_-(x, \lambda) , \quad (4.18)$$

into smooth 2×2 matrices ψ_+ and ψ_- which are holomorphic in $\lambda \in U_+$ and $\lambda \in U_-$, respectively. The splitting (4.18) can be considered as a solution of a parametric Riemann-Hilbert problem and the matrix-valued function ψ in (4.15)-(4.17) can be identified with $\psi_+(x, \lambda)$. Thus, one can apply various well-known methods (twistor approach, dressing method etc.) to solving vortex equations on Σ with the help of the Lax pair (4.16).

Existence of solutions. Note that, in general, there is a topological obstruction to the existence of N -vortex solution on a compact Riemann surface Σ [6]. In particular, for $g > 1$, from (3.26) and (4.13) we obtain that solutions of the vortex equations (3.23) exist if

$$N \leq 2(g-1) \quad (4.19)$$

since (4.13) fixes the area of Σ in terms of g and R^2 . Thus, for

$$N > 2(g-1) \quad (4.20)$$

the vortex equations on Σ written in the form (4.17) will have no solutions. However, one can rescale the metric on Σ , $g_{z\bar{z}} \rightarrow t^2 g_{z\bar{z}}$, and obtain inequalities

$$R_\Sigma^t = -\frac{2}{t^2 R^2} < -R_{S^2} = -\frac{2}{R^2} \quad \text{for } t^2 > 1, \quad (4.21)$$

$$N \leq 2t^2(g-1). \quad (4.22)$$

For any N the condition (4.22) can be satisfied for sufficiently large t and then the moduli space of vortices will be nonempty. However, this rescaling of metric on Σ leading to (4.21) brings us back to the case of pseudo-holomorphic bundles $\hat{\mathcal{E}}$ over \mathcal{Z} , i.e. to the case of nonintegrable almost complex structures. This is the price to be paid for having nonempty vortex moduli space when $N > 2(g-1)$. The value $N = 2(g-1)$ of the vortex number separates the holomorphic and nonholomorphic cases.

5 Conclusions

In this paper, we have shown the equivalence between vortex equations on a compact Riemann surface Σ , the Yang-Mills instanton equations on $\Sigma \times \mathbb{CP}^1$ and pseudo-Hermitian-Yang-Mills equations on the twistor space \mathcal{Z} of $\Sigma \times \mathbb{CP}^1$, summarized in the diagram (4.12). We have shown that in a special case, when the twistor geometry becomes integrable (holomorphic), the vortex equations on Σ appear as the commutator (4.17) of two auxiliary linear differential operators (with a ‘spectral’ parameter) having clear geometric sense. This brings us to a situation when one can, in principle, apply methods of integrable systems to finding solutions of vortex equations.

We considered vortices in the Abelian Higgs model. There are various non-Abelian generalizations of this model (see e.g. [30]-[35] and references therein). Results of this paper can be extended to the non-Abelian case. The simplest generalization can be obtained if one replaces the scalar function ϕ in (3.10) by $p \times q$ matrix and substitutes $1_p \otimes a$ and $-1_q \otimes a$ for $1 \otimes a$ and $-1 \otimes a$, respectively. We will consider this case in more detail elsewhere.

The constructions of this paper can also be generalized to \mathcal{N} -extended supersymmetric SDYM theory [36, 37] defined on $\Sigma \times \mathbb{CP}^1$ together with a reduction to supersymmetric vortex equations

on Σ . In this case vortices on Σ will correspond to supersymmetric Yang-Mills instantons in the background of gravitational instantons $\Sigma \times \mathbb{C}P^1$ of conformal (super)gravity. It is of interest since conformal supergravity interacting with Yang-Mills supermultiplets arises in the twistor string theory proposed recently [38, 39]. Vortices on Σ represent the simplest and very natural interaction of Yang-Mills instantons and gravitons in conformal gravity which cannot be reduced to Einstein gravity. These Yang-Mills/gravity configurations could be a good test background for calculation of open/closed twistor string amplitudes especially because the moduli spaces of Riemann surfaces and vortices are studied very well. It would be also interesting to look at our vortices from the cosmic string perspective.

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